Fibonacci, Lucas, Generalised Fibonacci and Golden section Formulae

Here are about 100 formula involving the Fibonacci numbers, the golden ratio and the Lucas numbers. This forms a major reference page for my Fibonacci Web site (http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/) where there are many more details, explanations and applications, with puzzles and tricks aimed at secondary school students and teachers as well as interested mathematical enthusiasts.

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Definitions and Notation

Beware of different golden ratio symbols used by different authors!
At this web site Phi is 1.618033... and phi is 0.618033.. but Vajda(see below) and Dunlap(see below) use a symbol for -0.618033.. .
Where a formula below (or a simple re-arrangement of it) occurs in either Vajda or Dunlap's book, the reference number they use is given. Dunlap's formulae are listed in his Appendix A3. Hoggatt's formula are from his "Fibonacci and Lucas Numbers" booklet. Full bibliographic details are at the end of this page.

As used here	**Vajda**	**Dunlap**	Description
floor(x)	[x]	trunc(x), not used for x<0	the nearest integer $\le$ x. **When x>0**, this is "the integer part of x" or "truncate x" i.e. delete any fractional part after the decimal point. 3=floor(3)=floor(3.1)=floor(3.9), -4=floor(-4)=floor(-3.1)=floor(-3.9)
round(x)	$[x + \frac{1}{2}]$	$\text{trunc}(x + \frac{1}{2})$	the nearest integer to x, equivalent to trunc(x+0.5) 3=round(3)=round(3.1), 4=round(3.9), -4=round(-4)=round(-3.9), -3=round(-3.1) 4=round(3.5), -3=round(-3.5)
ceil(x)	-	-	the nearest integer $\ge$ x. 3=ceil(3), 4=ceil(3.1)=ceil(3.9), -3=ceil(-3)=ceil(-3.1)=ceil(-3.9)
$\binom{n}{r}$	$\binom{n}{r}$	$\binom{n}{r}$	= n! ${}_nC_r$; n choose r; the element in row n

			r! (n – r)!	column r of Pascal's Triangle; the coefficient of x^r in $(1+x)^n$; the number of ways of choosing r objects from a set of n different objects. $n \geq 0$ and $r \geq 0$.

F(i) is the Fibonacci series and L(i) is the Lucas series.

Formula	**Refs**	Comments
F(0) = 0, F(1) = 1, F(n+2) = F(n + 1) + F(n)	-	Definition of the Fibonacci series
$F(-n) = (-1)^{n+1} F(n)$	Vajda-2, Dunlap-5	Extending the Fibonacci series 'backwards'
L(0) = 2, L(1) = 1, L(n + 2) = L(n + 1) + L(n)	-	Definition of the Lucas series
$L(-n) = (-1)^n L(n)$	Vajda-4, Dunlap-6	Extending the Lucas series 'backwards'
G(n + 2) = G(n + 1) + G(n)	Vajda-3, Dunlap-4	Definition of the Generalised Fibonacci series, G(0) and G(1) needed
$Phi = \frac{\sqrt{5} + 1}{2}$	Dunlap-63	Vajda and Dunlap use τ Phi and –phi are the roots of $x^2 = x + 1$
$phi = \frac{\sqrt{5} - 1}{2}$	Dunlap-65	Vajda uses $-\sigma$, and Dunlap uses $-\phi$ Beware! Dunlap occasionally uses ϕ to represent our phi = 0.61803.., but more frequently he uses ϕ to represent -0.618033..

Linear Relationships

Linear relationships involve only sums or differences of Fibonacci numbers or Lucas numbers or their multiples.

Two Fibonacci numbers

F(n + 3) + F(n) = 2 F(n + 2)	-
F(n + 3) – F(n) = 2 F(n + 1)	-
F(n + 4) + F(n) = 3 F(n + 2)	-
F(n + 4) – F(n) = L(n + 2)	-
F(n + 6) + F(n) = 2 L(n + 3)	-
F(n + 6) – F(n) = 4 F(n + 3)	-
F(n + 1) + F(n – 1) = L(n)	Vajda-6, Hoggatt-I8, Dunlap-14, Koshy-5.14
F(n) + 2 F(n – 1) = L(n)	(Dunlap-32)
F(n + 2) – F(n – 2) = L(n)	Vajda-7a, Dunlap-15, Koshy-5.15

F(n + 3) – 2 F(n) = L(n)	Dunlap possible correction for 31
F(n + 2) – F(n) + F(n – 1) = L(n)	Dunlap possible correction for 31

Two Lucas numbers

L(n – 1) + L(n + 1) = 5 F(n) L(n) + L(n + 2) = 5 F(n+1)	Vajda-5, Dunlap-13, Koshy-5.16
L(n) + L(n + 3) = 2 L(n + 2)	-
L(n) + L(n + 4) = 3 L(n + 2)	-
2 L(n) + L(n + 1) = 5 F(n + 1)	-
L(n – 2) – L(n + 2) = 5 F(n) L(n) – L(n+4) = 5 F(n + 2)	-
L(n + 3) – 2 L(n) = 5 F(n)	-

Sums with a Fibonacci and a Lucas number

F(n) + L(n) = 2 F(n + 1)	Vajda-7b, Dunlap-16
L(n) + 5 F(n) = 2 L(n + 1)	-
3 F(n) + L(n) = 2 F(n + 2)	Vajda-26, Dunlap-28
3 L(n) + 5 F(n) = 2 L(n + 2)	Vajda-27, Dunlap-29

Basic Golden Ratio Identities

Here Phi is Vajda's and Dunlap's tau (τ). phi here is Vajda's sigma (σ) and Dunlap's ϕ.

Phi phi = 1	Vajda page 51(3), Dunlap-65
Phi / phi = Phi + 1	-
Phi + phi = √5	-
phi / Phi = 1 – phi	-
Phi – phi = 1	-
Phi = phi + 1 = √5 - phi	-
phi = Phi – 1 = √5 - Phi	-
Phi^2 = Phi + 1	Vajda page 51(4), Dunlap-64
phi^2 + phi = 1	Vajda page 51(4), Dunlap-64
$Phi^{n+2} = Phi^{n+1} + Phi^n$	-
$phi^n = phi^{n+1} + phi^{n+2}$	-

Golden Ratio with Fibonacci and Lucas

Binet's Formula: where $\sqrt{5}=Phi-(-phi)$ $F(n) = \frac{Phi^n - (-phi)^n}{\sqrt{5}}$	Vajda-58, Dunlap-69, Hoggatt-page 11, Binet(1843), De Moivre(1718), Lamé(1844)
$L(n) = Phi^n + (-phi)^n$	Vajda-59, Dunlap-70
$F(n) = round\left(\frac{Phi^n}{\sqrt{5}}\right)$,if $n\geq 0$	Vajda-62, Dunlap-71 corrected
$L(n) = round(Phi^n)$,if $n\geq 2$	Vajda-63, Dunlap-72
$F(-n) = round\left(\frac{-(-phi)^{-n}}{\sqrt{5}}\right)$,if $n\geq 0$	-
$L(-n) = round((-phi)^{-n})$, $n\geq 3$	-
$F(-n) = (-1)^{n+1} round\left(\frac{Phi^n}{\sqrt{5}}\right)$,if $n\geq 0$	-
$L(-n) = round((-Phi)^n)$, $n\geq 3$	-
$F(n + 1) = round(Phi\ F(n))$,if $n\geq 2$	Vajda-64, Dunlap-73
$L(n + 1) = round(Phi\ L(n))$,if $n\geq 4$	Vajda-65, Dunlap-74
$F(n+1) - Phi\ F(n) = (-phi)^n$	Vajda-103b, Dunlap-75

Order 2 Fibonacci and Lucas Relationships

Order 2 means these formula have a term terms involving the product of at most 2 Fibonacci or Lucas numbers.

Fibonacci numbers

$F(2\ n) = F(n)^2 + 2\ F(n - 1)F(n)$	-
$F(2\ n + 1) = F(n + 1)^2 + F(n)^2$	Vajda-11, Dunlap-7, Lucas(1876)
$F(2\ n) = F(n + 1)^2 - F(n - 1)^2$	Lucas(1876)
$F(3\ n) = F(n + 1)^3 + F(n)^3 - F(n - 1)^3$	-
$F(n + 2)\ F(n - 1) = F(n + 1)^2 - F(n)^2$	Vajda-12, Dunlap-8
$F(n + 1)\ F(n - 1) - F(n)^2 = (-1)^n$	Vajda-29, Dunlap-9, Cassini's Formula(1680), Simson(1753) special case of Catalan's Identity with r=1
$F(n)^2 - F(n + r)F(n - r) = (-1)^{n-r}F(r)^2$	Catalan's Identity (1879)
$F(n)F(m + 1) - F(m)F(n + 1) = (-1)^mF(n - m)$	d'Ocagne's Identity, a special case of Vajda-9 with G=F

$F(n) = F(m) F(n + 1 - m) + F(m - 1) F(n - m)$	Dunlap-10
$F(n) F(n + 1) = F(n - 1) F(n + 2) + (-1)^{n-1}$	Vajda-20a special case: i:=1;k:=2;n:=n-1
$F(n + i) F(n + k) - F(n) F(n + i + k) = (-1)^n F(i) F(k)$	Vajda-20a=Vajda-18(corrected) with G:=H:=F
$F(n)^2 F(m + 1) F(m - 1) - F(m)^2 F(n + 1) F(n - 1) = (-1)^{n-1} F(m + n) F(m - n)$	Vajda-32

Two Lucas numbers

$L(2 n) = L(n)^2 - 2 (-1)^n$	-
$L(n + 2) L(n - 1) = L(n + 1)^2 - L(n)^2$	-
$L(n + 1) L(n - 1) - L(n)^2 = -5 (-1)^n$	-
$L(2 n) + 2 (-1)^n = L(n)^2$	Vajda-17c, Dunlap-12
$L(n + m) + (-1)^m L(n - m) = L(m) L(n)$	Vajda-17a, Dunlap-11

Fibonacci and Lucas Numbers

$F(2 n) = F(n) L(n)$	Vajda-13, Hoggatt-17, Koshy-5.13
$L(n + 1)^2 + L(n)^2 = 5 F(2 n + 1)$	-
$L(n + 1)^2 - 5 F(n) = L(2 n + 1)^2$	-
$L(2 n) - 2 (-1)^n = 5 F(n)^2$	Vajda-23, Dunlap-25
$F(n + 1) L(n) = F(2 n + 1) + (-1)^n$	Vajda-30, Vajda-31, Dunlap-27, Dunlap-30
$L(n + 1) F(n) = F(2 n + 1) - (-1)^n$	-
$F(2 n + 1) = F(n + 1) L(n + 1) - F(n) L(n)$	Vajda-14, Dunlap-18
$L(2 n + 1) = F(n + 1) L(n + 1) + F(n) L(n)$	-
$L(n)^2 - 2 L(2 n) = -5 F(n)^2$	Vajda-22, Dunlap-24
$5 F(n)^2 - L(n)^2 = 4 (-1)^{n+1}$	Vajda-24, Dunlap-26
$5 (F(n)^2 + F(n + 1)^2) = L(n)^2 + L(n + 1)^2$	Vajda-25
$5 F(2 n + 1)^2 = L(n)^2 + L(n + 1)^2$	Vajda-25a
$F(n) L(m) = F(n + m) + (-1)^m F(n - m)$	Vajda-15a, Dunlap-19

$L(n) F(m) = F(n + m) - (-1)^m F(n - m)$	Vajda-15b, Dunlap-20
$5 F(m) F(n) = L(n + m) - (-1)^m L(n - m)$	Vajda-17b, Dunlap-23
$2 F(n + m) = L(m) F(n) + L(n) F(m)$	Vajda-16a, Dunlap-21
$2 L(n + m) = L(m) L(n) + 5 F(n) F(m)$	-
$(-1)^m 2 F(n - m) = L(m) F(n) - L(n) F(m)$	Vajda-16b, Dunlap-22
$L(n + i) F(n + k) - L(n) F(n + i + k) = (-1)^{n+1} F(i) L(k)$	Vajda-19a
$F(n + i) L(n + k) - F(n) L(n + i + k) = (-1)^n F(i) L(k)$	Vajda-19b
$L(n + i) L(n + k) - L(n) L(n + i + k) = (-1)^{n+1} 5 F(i) F(k)$	Vajda-20b

Basic G Identities

G(i) is the General Fibonacci series. It has the same recurrence relation as Fibonacci and Lucas, namely **G(n+2) = G(n+1) + G(n) for all integers n (i.e. n can be negative)**, but the "starting values" of G(0) and G(1) can be specified. It therefore is a generalisation of both series and includes them both as special cases. Hoggatt and others use the letter H for series G.

e.g.

- If G(0)=0 and G(1)=1 we have 0,1,1,2,3,5,8,13,.. the Fibonacci series, i.e. G(0,1,i) = F(i);
- G(0)=2 and G(1)=1 gives 2,1,3,4,7,11,18,.. the Lucas series, i.e. G(2,1,i) = L(i);
- G(0)=1 and G(1)=1 gives 1,1,2,3,5,8,13,.. the Fibonacci series again but "moved left one place" i.e. G(1,1,i) = F(i+1).
- G(0,2,i) is 0,2,2,4,6,10,16,26,.. which is the Fibonacci series with all terms doubled, i.e. G(0,2,i) = 2 Fib(i).
- G(3,0,i) is 3,0,3,3,6,9,15,.. which is Fibonacci tripled and shifted right one place: G(3,0,i) = 3 F(i-1).
- G(3,2,i) is 3,2,5,7,12,19,31,.. is new - it is not a multiple of either the Fibonacci or Lucas series values.

$G(n + 2) = G(n + 1) + G(n)$	Vajda-3, Dunlap-4
$G(n) = G(0) F(n - 1) + G(1) F(n)$	-
$G(-n) = (-1)^n (G(0) F(n + 1) - G(1) F(n))$	-
$G(n + m) = F(m - 1) G(n) + F(m) G(n + 1)$	Vajda-8, Dunlap-33
$G(n - m) = (-1)^m (F(m + 1) G(n) - F(m) G(n + 1))$	Vajda-9, Dunlap-34
$L(m) G(n) = G(n + m) + (-1)^m G(n - m)$	Vajda-10a, Dunlap-35
$F(m) (G(n - 1) + G(n + 1)) = G(n + m) - (-1)^m G(n - m)$	Vajda-10b, Dunlap-36
$G(m) F(n) - G(n) F(m) = (-1)^{n+1} G(0) F(m - n)$	Vajda-21a
$G(m) F(n) - G(n) F(m) = (-1)^m G(0) F(n - m)$	Vajda-21b

Order 2 G Formulae

These formulae include terms which are a product of two G numbers either from the same G series of from two different G series i.e. with different index 0 and 1 values. Where the series may be different they are denoted G and H eg special cases include G = F (i.e. Fibonacci) and H = L (i.e. Lucas), or they could also be the same series G=H.

$G(n+i)\,H(n+k) - G(n)\,H(n+i+k) = (-1)^n\,(G(i)\,H(k) - G(0)\,H(i+k))$	Vajda-18 (corrected)
$G(n+1)\,G(n-1) - G(n)^2 = (-1)^n\,(G(1)^2 - G(0)\,G(2))$	Vajda-28
$\sqrt{5}\,G(n) = (G(1) + G(0)\,phi)\,Phi^n + (G(0)\,Phi - G(1))\,(-phi)^n$	Vajda-55/56, Dunlap-77

Fibonacci and Lucas Summations

These formulae involve a sum of Fibonacci or Lucas numbers.

$\sum_{i=0}^{n} F(i) = F(n+2) - 1$	Hoggatt-11, Lucas(1876)
$\sum_{i=0}^{n} L(i) = L(n+2) - 1$	Hoggatt-12
$\sum_{i=a}^{n} F(i) = F(n+2) - F(a+1)$	-
$\sum_{i=a}^{n} L(i) = L(n+2) - L(a+1)$	-
$\sum_{i=1}^{n} F(2i) = F(2n+1) - 1,\ n>=1$	Hoggatt-16, Lucas(1876)
$\sum_{i=1}^{n} L(2i) = L(2n+1) - 1$	-
$\sum_{i=1}^{n} F(2i-1) = F(2n),\ n>=1$	Hoggatt-15, Lucas(1876)
$\sum_{i=1}^{n} L(2i-1) = L(2n) - 2$	-
$\sum^{n} 2^{n-i}\,F(i-1) = 2^n - F(n+2)$	Vajda-37a(adapted), Dunlap-42(adapted)

i=1	
$\sum_{i=0}^{n} (-1)^i L(n-2i) = 2F(n+1)$	Vajda-97, Dunlap-54

Summations with fractions

$\sum_{i=0}^{\infty} \frac{F(i)}{2^i} = 2$	Vajda-60, Dunlap-51
$\sum_{i=0}^{\infty} \frac{L(i)}{2^i} = 6$	-
$\sum_{i=0}^{\infty} \frac{F(i)}{r^i} = \frac{r}{r^2-r-1}$	-
$\sum_{i=0}^{\infty} \frac{L(i)}{r^i} = 2 + \frac{r+2}{r^2-r-1}$	-
$\sum_{i=1}^{\infty} \frac{i F(i)}{2^i} = 10$	Vajda-61, Dunlap-52
$\sum_{i=1}^{\infty} \frac{i L(i)}{2^i} = 22$	-
$\sum_{i=1}^{\infty} \frac{1}{F(2^i)} = 4 - Phi = 3 - phi$	Vajda-77(corrected), Dunlap-53(corrected)

Order 2 summations

$\sum_{i=1}^{2n} F(i) F(i-1) = F(2n)^2$	Vajda-40, Dunlap-45
$\sum_{i=1}^{2n} L(i) L(i-1) = L(2n)^2 - 4$	-
$\sum_{i=1}^{2n+1} F(i) F(i-1) = F(2n+1)^2 - 1$	Vajda-42, Dunlap-47

$\sum_{i=1}^{2n+1} L(i)\,L(i-1) = L(2n+1)^2 - 5$	-
$\sum_{i=0}^{n-1} F(2i+1)^2 = \frac{F(4n) + 2n}{5}$	Vajda-95
$\sum_{i=0}^{n-1} L(2i+1)^2 = F(4n) - 2n$	Vajda-96
$\sum_{i=1}^{n} F(i)^2 = F(n)\,F(n+1)$	Vajda-45, Dunlap-5, Hoggatt-13, Lucas(1876), Koshy-77
$\sum_{i=1}^{n} L(i)^2 = L(n)\,L(n+1) - 2$	Hoggatt-14
$\sum_{i=1}^{2n-1} L(i)^2 = 5\,F(2n)\,F(2n-1)$	-
$5\sum_{i=0}^{n} F(i)\,F(n-i) \begin{cases} = (n+1)\,L(n) - 2\,F(n+1) \\ = n\,L(n) - F(n) \end{cases}$	Vajda-98, Dunlap-55
$\sum_{i=0}^{n} L(i)\,L(n-i) \begin{cases} = (n+1)\,L(n) + 2\,F(n+1) \\ = (n+2)\,L(n) + F(n) \end{cases}$	Vajda-99, Dunlap-56
$\sum_{i=0}^{n} F(i)\,L(n-i) = (n+1)\,F(n)$	Vajda-100, Dunlap-57
$\sum_{i=1}^{n} L(2i)^2 = F(4n+2) + 2n - 1$	Vajda page 70

General Summations

$\sum_{i=1}^{n} G(i) = G(n+2) - G(2)$	Vajda-33, Dunlap-38
$\sum^{n} G(i) = G(n+2) - G(a+1)$	-

i=a	
$\sum_{i=1}^{n} G(2i-1) = G(2n) - G(0)$	Vajda-34, Dunlap-37
$\sum_{i=1}^{n} G(2i) = G(2n+1) - G(1)$	Vajda-35, Dunlap-39
$\sum_{i=1}^{n} G(2i) - \sum_{i=1}^{n} G(2i-1) = G(2n-1) + G(0) - G(1)$	Vajda-36, Dunlap-40
$\sum_{i=1}^{n} 2^{n-i} G(i-1) = 2^{n-1}(G(0) + G(3)) - G(n+2)$	Vajda-37(variant), Dunlap-41(variant)
$\sum_{i=1}^{4n+2} G(i) = L(2n+1)\, G(2n+3)$	Vajda-38, Dunlap-43
$\sum_{i=1}^{2n} G(i)\, G(i-1) = G(2n)^2 - G(0)^2$	Vajda-39, Dunlap-44
$\sum_{i=1}^{2n+1} G(i)\, G(i-1) = G(2n+1)^2 - G(0)^2 - G(1)^2 + G(0)\, G(2)$	Vajda-41, Dunlap-46
$\sum_{i=1}^{n} G(i+2)\, G(i-1) = G(n+1)^2 - G(1)^2$	Vajda-43, Dunlap-48
$\sum_{i=1}^{n} G(i)^2 = G(n)\, G(n+1) - G(0)\, G(1)$	Vajda-44, Dunlap-49
$\sum_{i=0}^{\infty} \frac{G(a, b, i)}{r^i} = a + \frac{a + b r}{r^2 - r - 1}$	Stan Rabinowitz, "Second-Order Linear Recurrences" card, *Generating Function* special case (x=1/r, P=1, Q=-1)
$\sum_{i=0}^{\infty} \frac{i\, G(a, b, i)}{r^i} = \frac{r(b r^2 - 2 a r + b - a)}{(r^2 - r - 1)^2}$	-

$\sum_{i=1}^{n} \binom{n-i}{i-1} = F(n)$	-

$\sum_{i=0}^{\infty} \binom{n-i-1}{i} = F(n)$	Vajda-54(corrected), Dunlap-84(corrected)
$\sum_{i=0}^{n} \binom{n+1}{i+1} F(i) = F(2n+1) - 1$	Vajda-50, Dunlap-82
$\sum_{i=0}^{2n} \binom{2n}{i} F(2i) = 5^n F(2n)$	Vajda-69, Dunlap-85
$\sum_{i=0}^{2n} \binom{2n}{i} L(2i) = 5^n L(2n)$	Vajda-71, Dunlap-87
$\sum_{i=0}^{2n+1} \binom{2n+1}{i} F(2i) = 5^n L(2n+1)$	Vajda-70, Dunlap-86
$\sum_{i=0}^{2n+1} \binom{2n+1}{i} L(2i) = 5^{n+1} F(2n+1)$	Vajda-72, Dunlap-88
$\sum_{i=0}^{2n} \binom{2n}{i} F(i)^2 = 5^{n-1} L(2n)$	Vajda-73, Dunlap-89
$\sum_{i=0}^{2n} \binom{2n}{i} L(i)^2 = 5^n L(2n)$	Vajda-75, Dunlap-91
$\sum_{i=0}^{2n+1} \binom{2n+1}{i} F(i)^2 = 5^n F(2n+1)$	Vajda-74, Dunlap-90
$\sum_{i=0}^{2n+1} \binom{2n+1}{i} L(i)^2 = 5^{n+1} F(2n+1)$	Vajda-76, Dunlap-92
$\sum_{i=0}^{\infty} 5^i \binom{n}{2i+1} = 2^{n-1} F(n)$	Vajda-91
$\infty\ 5^i \binom{n}{2i} = 2^{n-1} L(n)$	Vajda-92

$\sum_{i=0}$	

With Generalised Fibonacci

$\sum_{i=0}^{n} \binom{n}{i} G(i) = G(2n)$	Vajda-47, Dunlap-80
$\sum_{i=0}^{n} \binom{n}{i} G(p-i) = G(p+n)$	Vajda-46, Dunlap-79
$\sum_{i=0}^{n} \binom{n}{i} G(p+i) = G(p+2n)$	Vajda-49, Dunlap-81
$\sum_{i=0}^{n} (-1)^i \binom{n}{i} G(n+p-i) = G(p-n)$	Vajda-51, Dunlap-83

References

S Vajda, **Fibonacci and Lucas numbers, and the Golden Section: Theory and Applications**, Halsted Press (1989).
This is a wonderful book! Unfortunately, it is now out of print. Vajda packs the book full of formulae on the Fibonacci numbers and Phi and the Lucas numbers. The whole book develops these formulae step by step, proving each from earlier ones or occasionally from scratch.

R A Dunlap, **The Golden Ratio and Fibonacci Numbers** World Scientific Press, 1997.
An introductory book strong on the geometry and natural aspects of the golden section and which does not dwell overmuch on the mathematical details. Beware - some of the formula in the Appendix are wrong! The formulae on this Web page are corrected versions and have been verified.

V E Hoggatt Jr Fibonacci and Lucas Numbers published by The Fibonacci Association, 1969 (Houghton Mifflin). A very good introduction to the Fibonacci and Lucas Numbers written by a founder of the Fibonacci Quarterly.

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